ANALYTICAL VELOCITY PROFILE IN TUBE FOR LAMINAR AND TURBULENT FLOW

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A new analytical formula of the velocity profile for both the laminar and turbulent flow in a tube with a circular cross-section will be introduced in this article. This formula is rather simple and easy to use. The advantage of this velocity profile is that one formula can be used for laminar and turbulent flow. This new formula will be compared with power law velocity profile and with the law of the wall also called as the log-law.

Keywords: *power-law, analytical velocity profile, vorticity, law of the wall, log-law, turbulent shear stress*

1. Introduction

The author is dealing with a fluid flow in a straight tube with a circular cross-section governed by the pressure gradient in this paper. Some fundamental ideas of the new analytical velocity profile derivation and its comparison with the power law velocity profile and with the log-law will be presented and discussed here.

The formula for the laminar velocity profile has to be mentioned first. The derivation of it is possible to find in every book dedicated to the fluid mechanics. The laminar velocity profile of the fluid flow governed by the pressure gradient is parabolic.

$$v = v_{(\max)} \left[1 - \left(\frac{r}{R}\right)^2\right] \tag{1}$$

where R is the tube radius, $v_{(\max)}$ is the maximal velocity or the centerline velocity of the velocity profile. It is assumed that there is only one velocity component in the tube axis direction. This velocity profile expression can be also rewritten as a function of average velocity $v_{(\mathrm{av})}$.

$$v = 2\, v_{(\mathrm{av})} \left[1 - \left(\frac{r}{R}\right)^2\right] . \tag{2}$$

This expression is more suitable for the practical use because the average velocity can be easy expressed from the flow rate and the radius of tube. Both these expressions can be rewritten to the dimensionless form.

$$\frac{v}{v_{(\max)}} = 1 - \left(\frac{r}{R}\right)^2 , \tag{3}$$

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$$\frac{v}{v_{(\mathrm{av})}} = 2\left[1-\left(\frac{r}{R}\right)^2\right] \ . \tag{4}$$

The expressions with the average velocity will be preferred in this paper.

In case of the turbulent flow it is more complicated to find some analytical solution. Many researches have been trying to find it. One of the well-known expressions of the turbulent velocity profile is the power law velocity profile. The power law velocity profile is for example mentioned in [3] and it is expressed this way.

$$v = v_{(\max)}\left[1-\frac{r}{R}\right]^{\frac{1}{n}} \ . \tag{5}$$

It is also possible to rewrite this formula as a function of the average velocity $v_{(\mathrm{av})}$ instead of the maximal velocity $v_{(\max)}$.

$$v = \frac{v_{(\mathrm{av})}}{2}\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)\left[1-\frac{r}{R}\right]^{\frac{1}{n}} \tag{6}$$

where n is a coefficient which is a function of the Reynolds number. It has to be determined on the basis of the experimental data. It is possible to find this dependence on a Reynolds' number in [3]. Some other expressions are mentioned in [2]. For example

$$n = 1+\sqrt[6]{\frac{\mathrm{Re}}{50}} \tag{7}$$

or n can be also expressed by this formula.

$$n = 1.03\,\ln(\mathrm{Re}) - 3.6 \ . \tag{8}$$

The value $n = 7$ is reasonable for many practical flow approximations as it is mentioned by Munson in [3].

This power law velocity profile has two fundamental discrepancies. First of them appears near the tube wall. It consists in the infinite derivative value on the wall which means that there is the infinite shear stress on the wall what represents infinite friction losses in the tube.

Second discrepancy is related to the velocity profile smoothness at the tube center. The first derivative of this formula is not smooth in the tube center. The utilizing of these formulas is then restricted because of these fundamental discrepancies.

The problem near the wall has to be solved a different way by using other empirical formulas which are valid only near the wall. The near wall region of the flow is called the boundary layer or the shear layer. The boundary layer is an area of the flow near the wall where magnitude of the friction (viscous) forces and dynamic forces are comparable. It can be also defined as the area near the wall with the not zero vorticity. It means that curl v is not zero. This boundary layer can be divided into three layers. The first of them is the one which is nearest to the wall. It is called viscous sub-layer. The second one is the transition area and the third one is the turbulent boundary layer.

It is necessary to define some quantities in order to be able to describe the velocity profile in the boundary layer.

$$v^* = \sqrt{\frac{\tau_{\mathrm{w}}}{\varrho}} \tag{9}$$

where v^* is the shear velocity, τ_w is shear stress on the wall, ϱ is fluid density. Then it is possible to define the dimensionless velocity (v^+) and the dimensionless distance from the wall (y^+).

$$v^+ = \frac{v}{v^*} \;, \tag{10}$$

$$y^+ = \frac{v^*\, y}{\nu} \tag{11}$$

where y is the distance from the wall.

The dimensionless velocity profile in viscous sublayer can be then expressed this way

$$v^+ = y^+ \;. \tag{12}$$

The dimensionless velocity profile in the turbulent boundary layer can be expressed by the log-law.

$$v^+ = \frac{1}{\kappa} \log(y^+) + B \;. \tag{13}$$

Different authors are using different values of the constants κ and B. For example the used coefficients are: $\kappa = 0.41$ and $B = 5.2$ in the book [1], $\kappa = 0.4$ and $B = 5$ in the book [2].

2. Introduction of a new velocity profile

The new velocity profile will be introduced after the previous brief overview of velocity profiles in a tube with circular cross-section. It is based on the vorticity density γ distribution over the tube cross-section. This vorticity induces the velocity. The vorticity density is closely related with vorticity vector Ω_i. The relationship between the induced velocity and the vorticity density is through the Biot-Sawart law. Biot-Sawart law is derived for a vortex filament element $\mathrm{d}s_j$. Vortex filament is represented by curve 's' with the circulation Γ around it. The situation is depicted in fig. 1. The Einstein summation convection will be used in all mathematical expressions.

$$\mathrm{d}v_i = \frac{\Gamma}{4\pi\, r^3}\, \varepsilon_{ijk}\, \mathrm{d}s_j\, r_k = \frac{\Gamma}{4\pi\, r^3}\, \varepsilon_{ijk}\, \mathrm{d}s_j\, (x'_k - x_k) \;. \tag{14}$$

For instance the Biot-Sawart law can be applied on the infinite straight vortex filament parallel to the x_3 axis with the constant circulation Γ. This situation is depicted in fig. 2.

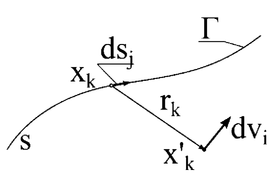

Fig. 1: Biot-Sawart law

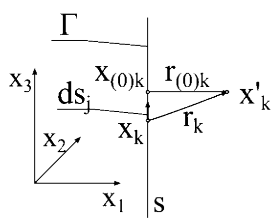

Fig. 2: Biot-Sawart law – the infinite straight vortex filament parallel with the axis x_3

The velocity induced by such vortex filament at the point x'_k can be expressed by the term.

$$v_i = \frac{\Gamma}{4\pi\, r_{(0)}^2}\, \varepsilon_{i3k}\, r_{(0)k} = \frac{\Gamma}{4\pi\, r_{(0)}^2}\, \varepsilon_{i3k}\, (x'_k - x_{(0)k}) \ . \tag{15}$$

Now the aim is to solve the velocity induced by the single circular vortex filament with the constant circulation around it. Unfortunately there is no analytical solution of this problem. The solution of it leads to the elliptical integrals. But the analytical solution exists for the case of infinite straight pipe.

It is necessary to introduce the linear vorticity density in the case of the flow inside the infinite straight pipe. The linear vorticity density is defined along the pipe axis, it means in direction x_1. The situation is depicted in fig. 3.

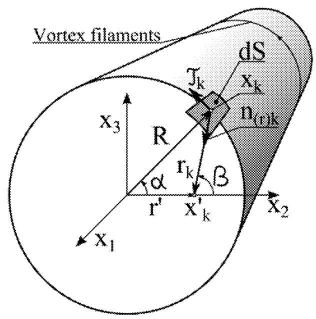

Fig.3: Circular vortex filaments aligned in the infinite length vortex tube

The relation between the linear vorticity density and the circulation is as follows.

$$\mathrm{d}\Gamma = \gamma\, \mathrm{d}x_1 \ . \tag{16}$$

Element of the circular vortex filament can be expressed this way

$$\mathrm{d}s_j = \tau_j\, \mathrm{d}s = \tau_j\, R\, \mathrm{d}\alpha \ . \tag{17}$$

It is possible to apply all the previous ideas into the Biot-Savart law.

$$\mathrm{d}v_i = \frac{\mathrm{d}\Gamma}{4\pi\, r^3}\, \varepsilon_{ijk}\, \mathrm{d}s_j\, (x'_k - x_k) = \frac{\gamma\, \mathrm{d}x_1}{4\pi\, r^3}\, \mathrm{d}s\, \varepsilon_{ijk}\, \tau_j\, (x'_k - x_k) \ . \tag{18}$$

Now it is possible to express the infinitesimal area of cylinder dS.

$$\mathrm{d}S = \mathrm{d}x_1\, \mathrm{d}s = \mathrm{d}x_1\, R\, \mathrm{d}\alpha \ . \tag{19}$$

The velocity induced by a vorticity closed inside the infinite tube with radius R consists of circular vortex filaments can be then expressed.

$$v_i = \int_S \frac{\varepsilon_{ijk}\, \gamma_j\, (x'_k - x_k)\, \mathrm{d}S}{2\pi\, r^3} \ . \tag{20}$$

The meaning of the quantities in the expression (10) is as follows; γ is the vorticity density vector, x'_k are the location coordinates of the induced velocity, x_k are location coordinates of the vortex filament element, $\mathrm{d}S$ is an infinitesimal area with constant vorticity density and constant tangential vector τ_j, r is the distance between the points x'_k and x_k. The area of integration S is an infinite cylindrical surface.

It is possible to find the analytical solution of this integral. The velocity induced by this vortex tube can be then expressed.

$$v_i = \gamma\, \varepsilon_{ijk}\, \tau_j\, n_{(\mathrm{r})k} \tag{21}$$

where the τ_j is a unit vector tangential to the vortex filament, $n_{(\mathrm{r})k}$ is the unit vector in r direction. The solution of a vector product $\varepsilon_{ijk}\, \tau_j\, n_{(\mathrm{r})k}$ is the unit vector in the tube axis direction.

Now it is necessary to express the velocity for the case that the vorticity density γ varies over cross-section it means that it is a function of radius inside of the tube. The variable radius will be marked as r (radius of the vortex tube). The radius of the tube will be marked R. It means that there will be a continuous distribution of the vorticity density over the cross-section. It will be assumed, that the vorticity density distribution will be a polynomial function of variable r.

$$\gamma = \sum_{n=0}^{N} A_{(n)}\, r^n \ . \tag{22}$$

The coefficients $A_{(n)}$ will be determined from the boundary conditions (slip condition on the wall), from the given flow rate through tube, and from the condition of smoothness. The velocity profile in a tube can be then expressed.

$$v = v_{(\mathrm{av})}\, \frac{N+3}{N+1} \left[1 - \left(\frac{r}{R}\right)^{N+1}\right] \ . \tag{23}$$

It is possible to compare this expression with the (1), (2) and (3). When $N = 1$ then it is the expression for the laminar velocity profile, for $N > 1$ it is turbulent velocity profile and for $N \to \infty$ it is a piston profile, it is the case of the infinite Reynolds number. The value of the power N can be expressed from the known pressure drop and flow rate in the tube.

$$N = \frac{R^2}{2\,\mu\, v_{(\mathrm{av})}}\, \frac{p_{(1)} - p_{(2)}}{L} - 3 \ . \tag{24}$$

3. Discusion

It is possible to compare the power-law velocity profile (6) with different expressions for the coefficient n and new velocity profile (23). The comparison is depicted in fig. 4. The comparison is done for $\mathrm{Re} = 10\,186$. The value of the coefficient n is expressed from the empirical formula (7) or (8). The power N in the case of new analytical velocity profile is evaluated from expression (24). The pressure drop is calculated for the case of the hydraulic smooth pipe. The friction coefficient λ can be then expressed by the formula.

$$\lambda = \frac{0.3164}{\sqrt[4]{\mathrm{Re}}} \ . \tag{25}$$

Consequently the pressure drop can be then expressed by the formula.

$$\frac{p_{(1)} - p_{(2)}}{L} = \lambda \, \frac{\varrho}{2\,R} \, \frac{v_{(\mathrm{av})}^2}{2} \; . \tag{26}$$

The comparison of the different analytical velocity profiles is depicted in the fig. 4. The velocity profiles are normalized by the average velocity $v_{(\mathrm{av})}$. The power-law velocity profile is drawn for the different values of the coefficient n. The case $n = 7$ is a most common value. The value $n = 3.48$ is evaluated from expression (7) and the value $n = 5.91$ is evaluated from the expression (8). The previous mentioned value of Reynolds number is used for evaluating of coefficient n. The radius R of the tube is 0.025 m. The laminar velocity profile is added to this set of velocity profiles for the comparison.

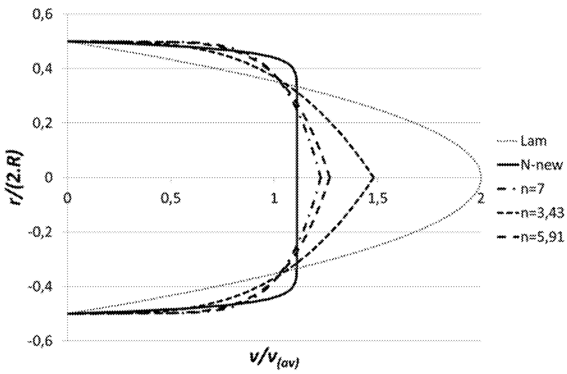

Fig.4: The comparison of velocity profiles

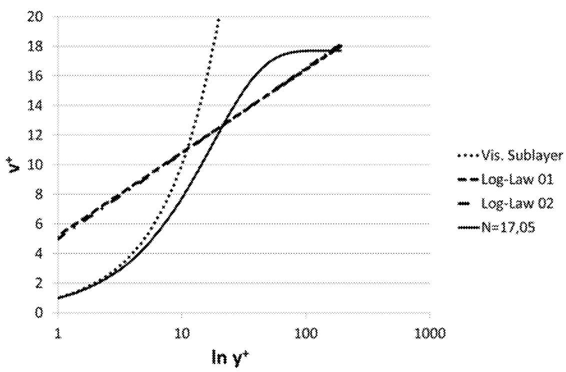

Fig.5: The Comparison of velocity profile near the wall with the log-law

It is apparent that all these turbulent velocity profiles are different. The power-law velocity profiles for different n have a problem near the wall, where the derivative is infinite. Next problem of it is in the tube center. First derivative is not smooth in the tube center. Probably most problematic discrepancy is the first one. The infinite derivative does not allow the study of the velocity profile near the wall because of the infinite shear stress at the wall. This means that the shear velocity is also infinite, in accordance with (9). It means that it is not possible to express the dimension-less velocity v^+ in the dependence on the dimension-les distance from the wall, in accordance to expression (11). So it means that is not possible to compare this power-law velocity profile with the log law near the wall.

The new velocity profile has no such problem near the wall and in the tube center like the power-law velocity profile. It means it is possible to compare this velocity profile near the wall with the log law. This comparison is done in fig. 5. The log-law is drawn there for two different sets of coefficients κ and B. The case 01 is drawn for $\kappa = 0.41$ and $B = 5.2$ in the book [1]. The case 02 is drawn for $\kappa = 0.4$ and $B = 5$ in the book [2].

The new velocity profile removes all the previous discrepancies of the power-law velocity profile and more over it offers other new features. The formula of the velocity profile (23) describes both extremes of the velocity profile the laminar flow and the flow with infinite Reynolds number. If the power $N = 1$ then the laminar velocity profile is obtained. If the value of N is going to the infinity then the piston profile is obtained.

The new velocity profile also has some problem. This problem is in the tube center. It consists in the velocity profile curvature. The value of second derivative in the tube center is zero. It means that the radius of curvature is infinite in the tube center. This is not true, but there is a chance to remove this discrepancy and the author is working on this problem.

It will be also interesting to draw the shear stresses at the tube cross-section. The total shear stress changes linearly. it can be divided into two parts the viscous shear stress and the turbulent (Reynolds) shear stress.

$$\tau = \tau_\mu + \tau_\mathrm{t} \ . \tag{25}$$

The viscous shear stress can be expressed from the velocity profile formula (23)

$$\tau_\mu = \mu \, \frac{\partial v}{\partial r} = -\mu \, v_{(\mathrm{av})} \, \frac{N+3}{R} \left(\frac{r}{R}\right)^N \ . \tag{26}$$

This viscous stress can be expressed in dimensionless form

$$\frac{\tau_\mu \, R}{\mu \, v_{(\mathrm{av})}} = -(N+3) \left(\frac{r}{R}\right)^N \ . \tag{27}$$

The turbulent shear stress can be obtained by subtracting the viscous stress from the total stress. The viscous and turbulent stresses over the cross-section for the laminar and two different turbulent velocity profiles are depicted in the fig. 6.

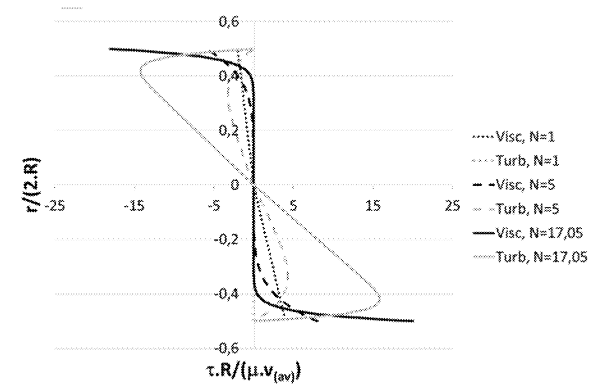

Fig.6: The viscosity and turbulent shear stresses

4. Conclusion

The new analytical velocity profile for both the laminar and the turbulent flow in the infinite length tube has been introduced in this article. This profile avoids some discrepancies of the power law velocity profile. This velocity profile offers new interesting features as the study of the near wall velocity profile. It rather good approximates the log-law velocity profile near the wall. But it still has some problems which have to be solved. The main problem is that there is the infinite curvature radius in the tube center. The author is working on it to avoid this problem. At the end it is also necessary to compare this velocity profile with an experimental data.

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Nomenclature

Symbol	Units	Description
$A(n)$	Varies	Polynomial coefficients
L	[m]	Length – distance between the pressure location measuring
n	–	Coefficient for the power law velocity profile
N	–	Order of the polynomial, power
$n_{(\mathrm{r})k}$	–	Unit vector in direction of r_k vector
Δp	[Pa]	Pressure difference
$p_{(1)}, p_{(2)}$	[Pa]	Pressure at location 1 or 2 respectively
Q	[m]	Unit flow rate. Flow rate between two parallel plates with 1 m width
r, R	[m]	Radius
Re	[–]	Reynolds number
r_k	[m]	Vector between points x_k and x'_k
$r_{(0)}$	[m]	Distance of point x'_k from straight vortex filament
$v_{(\mathrm{av})}$	$[\mathrm{m\,s^{-1}}]$	Average velocity between two parallel plates
v, v_i	$[\mathrm{m\,s^{-1}}]$	Component of velocity in x direction
$v_{(\max)}$	$[\mathrm{m\,s^{-1}}]$	Maximal velocity component in x direction
v^*	$[\mathrm{m\,s^{-1}}]$	Shear velocity
v^+	[–]	Dimensionless velocity near the wall
v_i	$[\mathrm{m\,s^{-1}}]$	Velocity vector
v_1, v_2, v_3	$[\mathrm{m\,s^{-1}}]$	Components of velocity vector
x_1, x_2, x_3	[m]	Coordinates of location
x'_k	[m]	Coordinates of the induced velocity location
$x_{(0)k}$	[m]	Coordinates of point x'_k projected onto the vortex filament
y	[m]	Coordinate y, or distance from wall
y^+	[–]	Dimensionless distance from the wall
Γ	$[\mathrm{m^2\,s^{-1}}]$	Circulation
γ	$[\mathrm{m\,s^{-1}}]/[\mathrm{s^{-1}}]$	Linear/planar vorticity density
ε_{ijk}	[–]	Levi-Civita tensor
λ	–	Friction coefficient
μ	[Pa s]	Dynamic viscosity
ν	$[\mathrm{m^2\,s^{-1}}]$	Kinematic viscosity
ϱ	$[\mathrm{kg\,m^{-3}}]$	Density
τ_j	[–]	Unit tangential vector
τ_{t}	[Pa]	Turbulent (Reynolds) shear stress
τ_{w}	[Pa]	Shear stress on the wall
τ_μ	[Pa]	Viscous shear stress

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